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Equivalence Theorem on Weighted Simultaneous L_p-Approximation by the Method of Kantorovič Operators

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The aim of the present paper is to prove new direct and converse results on weighted simultaneous approximation by the method of classical Kantorovič operators in the L_p-metric using the weighted Ditzian and Totik modulus of smoothness. We will obtain with a direct technique for integrable functions the complete equivalence characterization of weighted simultaneous approximation by this method $(K_n)_{n\in\mathbb{N}}$. One of the main tools and crucial estimates managing the converse and equivalence results of the simultaneous behaviour is given by a direct modified Voronovskaja theorem which uses the third order weighted modulus of smoothness.

1. INTRODUCTION

For functions $f \in L_p[0, 1]$, $1 \le p \le \infty$, (with $C[0, 1]$ for $p = \infty$), the nth classical Kantorovič operator K_n, $n \in \mathbb{N}$ [9], is defined by

$$K_n(f; x) := (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\,dt,$$
$$x, t \in [0, 1],\ n \in \mathbb{N}, \tag{1.1}$$

with

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Direct and converse theorems on the global rate of L_p-approximation by the method $(K_n)_{n\in\mathbb{N}}$ have been treated in various papers. Without claim of completeness we mention [4, 5, 8, 12–19].

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Our aim is to characterize the global rate of weighted simultaneous L_p-approximation $\|\varphi^{2r}(K_n f - f)^{(2r)}\|_p$, $\varphi(x) = \sqrt{x(1-x)}$, $r \in \mathbb{N}$, by the method $(K_n)_{n\in\mathbb{N}}$ to functions $f \in L_p[0,1]$, $1 \le p < \infty$ with $f^{(2r)} \in L_p[0,1]$ by making use of the special weighted second order Ditzian–Totik modulus $\omega_\varphi^2(f^{(2r)};t)_{\varphi^{2r},p}$ (cf. [3]). Essential for the proofs is for functions f with $\varphi^{2r} f \in L_p[0,1]$ the equivalence of $\omega_\varphi^s(f;t)_{\varphi^{2r},p}$, $s \in \mathbb{N}$, both to the weighted K-functional

$$K_\varphi^s(f;t^s)_{\varphi^s,p} := \inf\left\{\|\varphi^{2r}(f-g)\|_p + t^s\|\varphi^{2r+s}g^{(s)}\|_p \middle| \varphi^{2r}g, \varphi^{2r+s}g^{(s)} \in L_p[0,1]\right\} \tag{1.2}$$

and to the modified weighted K-functional

$$\overline{K}_\varphi^s(f;t^s)_{\varphi^{2r},p} := \inf\left\{\|\varphi^{2r}(f-g)\|_p + t^s\|\varphi^{2r+s}g^{(s)}\|_p + t^{2s}\|\varphi^{2r}g^{(s)}\|_p \middle| \varphi^{2r}g, \varphi^{2r}g^{(s)} \in L_p[0,1]\right\}, \tag{1.3}$$

(cf. [3, 6]), i.e.,

$$\omega_\varphi^s(f;t)_{\varphi^{2r},p} \sim K_\varphi^s(f;t^s)_{\varphi^{2r},p} \sim \overline{K}_\varphi^s(f;t^s)_{\varphi^{2r},p}. \tag{1.4}$$

2. Definitions and Auxiliary Results

If $f \in L_p[0,1]$, $1 \le p \le \infty$ with $f^{(r)} \in L_p[0,1]$, $n \ge r$, $x \in [0,1]$. Then

$$K_n^{(r)}(f;x) := (K_n(f;x))^{(r)} = \frac{(n+1)!}{(n-r)!}\sum_{k=0}^{n-r} p_{n-r,k}(x)\int_0^{1/(n+1)}\cdots \times \int_0^{1/(n+1)}\int_{k/(n+1)}^{(k+1)/(n+1)} f^{(r)}(t+u_1+\cdots+u_r)\,dt\,du_1\cdots du_r. \tag{2.1}$$

For convenience we will work occasionally with operators $K_{n,r}$ given for $g \in L_p[0,1]$, $1 \le p \le \infty$, $n \ge r$, $x \in [0,1]$ by

$$K_{n,r}(g;x) := \frac{(n+1)!}{(n-r)!}\sum_{k=0}^{n-r} p_{n-r,k}(x)\int_0^{1/(n+1)}\cdots \times \int_0^{1/(n+1)}\int_{k/(n+1)}^{(k+1)/(n+1)} g(t+u_1+\cdots+u_r)\,dt\,du_1\cdots du_r. \tag{2.2}$$

Evidently,

$$K_n^{(r)}(f;x) = K_{n,r}(f^{(r)};x), \qquad n \geq r, \tag{2.3}$$

if $f \in L_p[0,1]$ with $f^{(r)} \in L_p[0,1]$.

LEMMA 2.1. *For the functions $\Omega_{i,u}(t) := (t-u)^i$, $i \in \mathbb{N}$, we obtain*

$$K_n^{(r)}(\Omega_{i,u};x)_{u=x} = 0 \qquad \text{for } i \in \{0,1,\ldots,r-1\}, \tag{2.4}$$

$$K_n^{(r)}(\Omega_{r,u};x)_{u=x} = r!\frac{n!}{(n-r)!(n+1)^r}, \tag{2.5}$$

$$K_n^{(r)}(\Omega_{r+1,u};x)_{u=x} = (r+1)!\frac{n!}{(n-r)!(n+1)^r}\left(\frac{\frac{r+1}{2}(1-2x)}{n+1}\right) \tag{2.6}$$

and

$$K_n^{(r)}(\Omega_{r+2,u};x)_{u=x} = \frac{(r+2)!}{2}\frac{n!}{(n-r)!(n+1)^r} \times\left(\frac{a-(r^2+3r-n+1)x(1-x)}{(n+1)^2}\right), \tag{2.7}$$

where $a := (3r+4)(r+1)/12$.
Moreover, for $n, r, s \in \mathbb{N}$, and $x \in E_n := [1/n, 1-(1/n)]$, we have

$$|K_n^{(2r)}(\Omega_{2s,u};x)_{u=x}| \leq C\sum_{i=0}^{2r}\sum_{j=0}^{s+(i/2)}\left(\frac{\varphi^2(x)}{n}\right)^{s-r-j} n^{-2j}, \qquad (s \geq r), \tag{2.8}$$

and thus for $m \in \mathbb{N}$

$$\begin{aligned}|K_n^{(2r)}(\Omega_{2r+m,u};x)_{u=x}| &\leq C(\varphi^2(x)n^{-1})^{m/2}\\ &\leq Cn^{-m/2}(\varphi^2(x)+n^{-1})^{m/2},\end{aligned} \tag{2.9}$$

where $C := C(r)$ is independent of k and x.

Proof. Elementary calculations. ∎

Throughout this paper C will denote a positive constant not necessarily the same at each occurrence.

For the investigation of weighted direct and converse results in Section 3 we use the following preliminary results (called Bernstein–Markov-type inequalities), which are of importance in themselves [3, (9.4.1) and (9.7.7)]:

$$\left\|\varphi^{2r}(K_n f)^{(2r)}\right\|_p \le Cn^r\|f\|_p, \qquad f \in L_p[0,1] \tag{2.10}$$

and

$$\left\|\varphi^{2r}(K_n f)^{(2r)}\right\|_p \le C\|\varphi^{2r}f^{(2r)}\|_p, \qquad f \in L_p[0,1] \quad \text{with} \quad f^{(2r)} \in L_p[0,1], \tag{2.11}$$

Here $1 \le p \le \infty$, $\varphi(x) = \sqrt{x(1-x)}$ and C is a constant independent of n.

For the higher derivatives of order $(2r + 2s)$ we state analogous results in

LEMMA 2.2. *Let $1 \le p \le \infty$, $\varphi(x) = \sqrt{x(1-x)}$ and $n, s, r \in \mathbb{N}$. Then we have for $n \ge 2r + 2s$*

$$\left\|\varphi^{2r+2s}(K_n f)^{(2r+2s)}\right\|_p \le C\|\varphi^{2r+2s}f^{(2r+2s)}\|_p, \qquad f \in L_p[0,1] \quad \textit{with} \quad \varphi^{2r+2s}f^{(2r+2s)} \in L_p[0,1] \tag{2.12}$$

and

$$\left\|\varphi^{2r+2s}(K_n f)^{(2r+2s)}\right\|_p \le Cn^s\|\varphi^{2r}f^{(2r)}\|_p, \qquad f \in L_p[0,1] \quad \textit{with} \quad \varphi^{2r}f^{(2r)} \in L_p[0,1], \tag{2.13}$$

where C is a constant independent of n.

Proof. Fix $n \ge 2r + 2s$. Let $P_{[\sqrt{n}]}$ be the best approximating polynomial of degree $[\sqrt{n}]$ to $f \in L_p[0,1]$. Then we obtain with (2.10)

$$\begin{aligned}
&\left\|\varphi^{2r+2s}(K_n f)^{(2r+2s)}\right\|_p \\
&\quad \le \left\|\varphi^{2r+2s}K_n^{(2r+2s)}(f - P_{[\sqrt{n}]})\right\|_p + \left\|\varphi^{2r+2s}K_n^{(2r+2s)}P_{[\sqrt{n}]}\right\|_p \\
&\quad \le Cn^{r+s}\left\|f - P_{[\sqrt{n}]}\right\|_p + \left\|\varphi^{2r+2s}K_n^{(2r+2s)}P_{[\sqrt{n}]}\right\|_p.
\end{aligned} \tag{2.14}$$

Moreover, using the estimate on weighted polynomial approximation

[3, (8.1.3)] and (2.11), we have

$$\begin{aligned}\left\|\varphi^{2r+2s}K_n^{(2r+2s)}P_{[\sqrt{n}]}\right\|_p &= \left\|\varphi^{2r+2s}\left(K_n^{(2r)}P_{[\sqrt{n}]}\right)^{(2s)}\right\|_p \\ &\le C\left(\left[\sqrt{n}\right]\right)^{2s}\left\|\varphi^{2r}K_n^{(2r)}P_{[\sqrt{n}]}\right\|_p \\ &\le Cn^s\left\|\varphi^{2r}P_{[\sqrt{n}]}^{(2r)}\right\|_p.\end{aligned}$$

For the first term on the right-hand side of (2.14) we employ two results on best polynomial approximation on [0, 1], [3, (7.2.2) and (7.3.1)], namely,

$$\left\|f - P_{[\sqrt{n}]}\right\|_p \le C\omega_\varphi^{2r}(f; n^{-1/2})_p \quad \text{and} \quad \left\|\varphi^{2r}P_{[\sqrt{n}]}{}^{(2r)}\right\|_p \le Cn^r\omega_\varphi^{2r}(f; n^{-1/2})_p,$$

leading to

$$\begin{aligned}\|\varphi^{2r+2s}K_n^{(2r+2s)}f\|_p &\le Cn^{r+s}\left\{\left\|f - P_{[\sqrt{n}]}\right\|_p + n^{-r}\left\|\varphi^{2r}P_{[\sqrt{n}]}{}^{(2r)}\right\|_p\right\} \\ &\le Cn^{r+s}\omega_\varphi^{2r}(f; n^{-1/2})_p.\end{aligned}$$

Finally, we have to show that for $f \in L_p[0,1]$ with $\varphi^{2r}f^{(2r)} \in L_p[0,1]$ the following inequality is true

$$\omega_\varphi^{2r}(f; n^{-1/2})_p \le Cn^{-r}\|\varphi^{2r}f^{(2r)}\|_p.$$

Therefore we use the equivalence

$$\omega_\varphi^{2r}(f; n^{-1/2})_p \sim K_\varphi^{2r}(f; n^{-r})_p$$

with

$$K_\varphi^{2r}(f; n^{-r})_p = \inf_g\left\{\|f - g\|_p + n^{-r}\|\varphi^{2r}g^{(2r)}\|_p\right\},$$

where we take the infimum over all $g \in L_p[0,1]$ with $\varphi^{2r}g^{(2r)} \in L_p[0,1]$. Since f itself meets these requirements we obtain

$$K_\varphi^{2r}(f; n^{-r})_p \le n^{-r}\|\varphi^{2r}f^{(2r)}\|_p,$$

which proves the lemma. ∎

On account of relation (2.3) we are able to transfer Lemma 2.2 into

COROLLARY 2.3. *Let* $1 \le p \le \infty$, $\varphi(x) = \sqrt{x(1-x)}$ *and* $n, r, s \in \mathbb{N}$. *Then for* $n \ge 2r + 2s$ *there is*

$$\left\|\varphi^{2r+2s}(K_{n,2r}g)^{(2s)}\right\|_p \le C\|\varphi^{2r+2s}g^{(2s)}\|_p,$$
$$g \in L_p[0,1] \quad \textit{with} \quad \varphi^{2r+2s}g^{(2s)} \in L_p[0,1] \quad (2.15)$$

and

$$\left\|\varphi^{2s+2r}\left(K_{n,2r}g\right)^{(2s)}\right\|_p \le Cn^s\|\varphi^{2r}g\|_p, \qquad g \in L_p[0,1], \quad (2.16)$$

where C is a constant independent of n.

With this preliminary discussion we can prove the following weighted simultaneous estimate with the modulus of smoothness.

LEMMA 2.4. *Let $f \in L_p[0,1]$ with $f^{(2r)} \in L_p[0,1]$, $1 \le p \le \infty$, $\varphi(x) = \sqrt{x(1-x)}$ and $n, r, s \in \mathbb{N}$. Then for $n \ge 2r + 2s$*

$$\begin{aligned}\left\|\varphi^{2r+2s}\left(K_n f\right)^{(2r+2s)}\right\|_p &= \left\|\varphi^{2r+2s}\left(K_{n,2r}f^{(2r)}\right)^{(2s)}\right\|_p \\ &\le Cn^s\omega_\varphi^{2s}\left(f^{(2r)}; n^{-1/2}\right)_{\varphi^{2r},p}, \qquad (2.17)\end{aligned}$$

where C is a constant independent of n.

Proof. For every g with $\varphi^{2r}g, \varphi^{2r+2s}g^{(2s)} \in L_p[0,1]$ we obtain with (2.15) and (2.16)

$$\begin{aligned}&\left\|\varphi^{2r+2s}\left(K_{n,2r}f^{(2r)}\right)^{(2s)}\right\|_p \\ &\le \left\|\varphi^{2r+2s}\left(K_{n,2r}\left(f^{(2r)} - g\right)\right)^{(2s)}\right\|_p + \left\|\varphi^{2r+2s}\left(K_{n,2r}g\right)^{(2s)}\right\|_p \\ &\le Cn^s\left\{\left\|\varphi^{2r}\left(f^{(2r)} - g\right)\right\|_p + n^{-s}\|\varphi^{2r+2s}g^{(2s)}\|_p\right\}.\end{aligned}$$

Taking on the right-hand side the infimum over all g, we prove this theorem by using the equivalence relation between the weighted K-functional and the corresponding modulus of smoothness (1.4). ∎

The following lemma gives us a useful bound for the weighted norm of $K_n^{(2r)}R_s$, where

$$R_s(f;t;x) = \frac{1}{(s-1)!}\int_x^t (t-u)^{s-1} f^{(s)}(u)\,du, \qquad x, t \in (0,1)$$

denotes the integral remainder in the Taylor expansion of a function $f \in L_p[0,1]$ with $f^{(s)} \in L_p[0,1]$, $1 \le p \le \infty$.

LEMMA 2.5. *Let $g \in L_p[0,1]$ with $g^{(2r+m)} \in L_p[0,1]$, $1 \le p \le \infty$, $\varphi(x) = \sqrt{x(1-x)}$, $m, r \in \mathbb{N}$. Then*

$$\left\|\varphi^{2r}K_n^{(2r)}\left(R_{2r+m}g\right)\right\|_p^{E_n} \le Cn^{-m/2}\left\|\varphi^{2r}\left(\varphi^2 + n^{-1}\right)^{m/2}g^{(2r+m)}\right\|_p, \quad (2.18)$$

where the constant C is independent of n.

Remark 2.6. For technical reasons the additional term n^{-1} in (2.18) is useful in a similar way as it has been in (2.9). This gives the estimate

$$\|\varphi^{2r}K_n^{(2r)}(R_{2r+m}g)\|_p^{E_n} \leq Cn^{-m/2}\|\varphi^{2r+m}g^{(2r+m)}\|_p. \tag{2.19}$$

Before being able to prove Lemma 2.5, we need the following two lemmas and a corollary.

LEMMA 2.7. *Let $x, t \in [0,1]$ and $f, f_1, Q, Q_1 \in L_p[0,1]$. Then we have for $n \geq r$ the symmetry property*

$$K_n^{(2r)}(Q(t)h(t); x) = K_n^{(2r)}(Q(t)h(t); 1-x), \tag{2.20}$$

where Q: $[0,1] \to \mathbb{R}$ is a function with $Q(t) = Q(1-t)$ and $h(t) := (f(t) + f(1-t))/2$.

Moreover, the function

$$Z(x) := Q_1(x)h_1(x)K_n^{(2r)}(Q(t)h(t); x), \tag{2.21}$$

where Q, Q_1: $[0,1] \to \mathbb{R}$ with $Q(t) = Q(1-t)$, $Q_1(t) = Q_1(1-t)$, $h(t) := (f(t) + f(1-t))/2$, $h_1(t) := (f_1(t) + f_1(1-t))/2$, has the symmetry property for $x \in [0,1]$:

$$Z(x) = Z(1-x). \tag{2.22}$$

Proof. For the Kantorovič polynomials (1.1) there holds

$$K_n(f(1-t); 1-x) = K_n(f(t); x), \qquad n \in \mathbb{N}$$

and with $Q(t) = Q(1-t)$ we have

$$\begin{aligned} K_n(f(t)Q(t); x) &= K_n(f(1-t)Q(1-t); 1-x) \\ &= K_n(f(1-t)Q(t); 1-x) \end{aligned}$$

and

$$K_n(f(1-t)Q(t); x) = K_n(f(t)Q(t); 1-x).$$

On account of the linearity of K_n and the definition of h we obtain

$$K_n(h(t)Q(t); x) = K_n(h(t)Q(t); 1-x), \qquad n \in \mathbb{N}. \tag{2.23}$$

From this equation (2.20) and (2.22) follow immediately. ∎

Remark 2.8. If the assumptions of Lemma 2.7 are satisfied, then

$$\|h\|_p \leq \|f\|_p, \qquad \|hQ\|_p \leq \|fQ\|_p.$$

An example of a function with the requested property of $Q(t)$ is

$$Q(t) := \varphi^{2m}(t)(\varphi^{2k}(t) + n^{-1}),$$
$$\varphi(t) = \sqrt{t(1-t)}, \qquad k, m, n \in \mathbb{N},\ t \in [0,1].$$

For the proof of Lemma 2.5 we also need an estimate for the image of the function $f(t) := t^{-r}$.

LEMMA 2.9. *Let $t \in (0,1]$ and $n, r \in \mathbb{N}$,. Then for $n \geq r$*

$$K_{n,r}(t^{-r}; x) \leq Cx^{-r}, \qquad x \in (0,1], \tag{2.24}$$

with a constant C independent of n.

Proof. For $t \in (0,1]$ we have

$$K_{n,r}(t^{-r}; x) = \sum_{k=0}^{n-r} S_k(t^{-r}; x), \tag{2.25}$$

where

$$S_k(t^{-r}; x) := \frac{(n+1)!}{(n-r)!} p_{n-r,k}(x) \int_0^{1/(n+1)} \cdots$$
$$\times \int_0^{1/(n+1)} \int_{k/(n+1)}^{(k+1)/(n+1)} (t + u_1 + \cdots + u_r)^{-r}\, dt\, du_1 \cdots du_r,$$

for $k \in \{0, 1, \ldots, n-r\}$.

Since $h(t, u_1, \ldots, u_r) := (t + u_1 + \cdots + u_r)^{-r}$ is monoton decreasing in each of its arguments, we easily obtain for $k \in \{1, \ldots, n-r\}$

$$S_k(t^{-r}; x) \leq \frac{n!}{(n-r)!(n+1)^r} p_{n-r,k}(x) \left(\frac{k}{n+1}\right)^{-r} \tag{2.26}$$

and for $k = 0$ by a limit process

$$S_0(t^{-r}; x) \leq Cx^{-r} p_{n,r}(x). \tag{2.27}$$

Observing that

$$p_{n-r,k}(x) = \frac{(n-r)!(k+r)!}{n!k!} x^{-r} p_{n,k+r}(x), \tag{2.28}$$

and

$$\frac{(k+r)!}{k!} = \prod_{i=1}^{r} (k+i) \leq Ck^r \tag{2.29}$$

we have

$$K_{n,r}(t^{-r};x) \leq Cx^{-r}p_{n,r}(x) + x^{-r}\sum_{k=1}^{n-r}\frac{(k+r)!}{k!}p_{n,k+r}(x)k^{-r}$$
$$\leq Cx^{-r}\sum_{k=0}^{n-r}p_{n,k+r}(x) \leq Cx^{-r},$$

which concludes the proof. ∎

With Lemma 2.9 and the symmetry property in Lemma 2.7 we derive

COROLLARY 2.10. *Let $x, t \in [0,1)$ and $n, r \in \mathbb{N}$, $n \geq 2r$. Then*

$$K_{n,2r}\big((1-t)^{-2r};x\big) = K_{n,2r}(t^{-2r};1-x) \leq C(1-x)^{-2r}, \quad (2.30)$$

where the constant C is independent of n.

With the symmetry property in (2.20) and (2.22), Remark 2.8, and Corollary 2.10 we now have helpful tools for the proof of Lemma 2.5.

The left-hand side of (2.18)

$$\|\varphi^{2r}K_n^{(2r)}(R_{2r+m}g)\|_p^{E_n}$$
$$= \left\{\int_{E_n}\left|\varphi^{2r}(x)K_n^{(2r)}\left(\frac{1}{(2r+m-1)!}\int_x^t (t-u)^{2r+m-1}\right.\right.\right.$$
$$\left.\left.\left.\times g^{(2r+m)}(u)\,du;x\right)\right|_{u=x}\right|^p dx\right\}^{1/p}, \quad (2.31)$$

will be estimated for $p = 1$ and $p > 1$ separately. We start with $p > 1$ and distinguish the two cases

$$0 < t + u_1 + \cdots + u_{2r} \leq x < 1$$

and

$$0 < x \leq t + u_1 + \cdots + u_{2r} < 1.$$

Case 1. From $0 < t \leq t + u_1 + \cdots + u_{2r} \leq u \leq x < 1$ we find

$$0 \leq \frac{u - (t + u_1 + \cdots + u_{2r})}{u} \leq \frac{x - (t + u_1 + \cdots + u_{2r})}{x}. \quad (2.32)$$

For $n \in \mathbb{N}$ we have $n^{-1}/x \leq n^{-1}/u$, $1 - x \leq 1 - u$ and therefore

$$\frac{u}{\varphi^2(u) + n^{-1}} \leq \frac{x}{\varphi^2(x) + n^{-1}}. \tag{2.33}$$

From $\varphi^2(x)/x \leq \varphi^2(u)/u$ we obtain for $r \in \mathbb{N}$

$$\varphi^{2r}(x) \leq \varphi^{2r}(u)\left(\frac{u}{x}\right)^{-r}. \tag{2.34}$$

Combining (2.32), (2.33), and (2.34), we have

$$\begin{aligned}
&\varphi^{2r}(x)\big(u - (t + u_1 + \cdots + u_{2r})\big)^{2r+m-1} \\
&\quad \leq \varphi^{2r}(x)\big(x - (t + u_1 + \cdots + u_{2r})\big)^{2r+m-1}\left(\frac{u}{x}\right)^{2r+m-1} \\
&\quad \leq \varphi^{2r}(u)\big(x - (t + u_1 + \cdots + u_{2r})\big)^{2r+m-1} \\
&\qquad \times \left(\frac{\varphi^2(u) + n^{-1}}{\varphi^2(x) + n^{-1}}\right)^{m/2}\left(\frac{u}{x}\right)^{r+(m/2)-1},
\end{aligned}$$

giving for $r + (m/2) \geq 1$, i.e., $m \in \mathbb{N}$, the estimate

$$\begin{aligned}
0 &\leq \varphi^{2r}(x)\big(u - (t + u_1 + \cdots + u_{2r})\big)^{2r+m-1} \\
&\leq \varphi^{2r}(u)\big(x - (t + u_1 + \cdots + u_{2r})\big)^{2r+m-1}\left(\frac{\varphi^2(u) + n^{-1}}{\varphi^2(x) + n^{-1}}\right)^{m/2}.
\end{aligned} \tag{2.35}$$

Case 2. Here we have $0 < x \leq u \leq t + u_1 + \cdots + u_{2r} < 1$. Using $n^{-1}/(1 - x) \leq n^{-1}/(1 - u)$ and therefore

$$1\Big/\left(u + \frac{n^{-1}}{1 - u}\right) \leq 1\Big/\left(x + \frac{n^{-1}}{1 - x}\right),$$

we obtain

$$\frac{1 - u}{\varphi^2(u) + n^{-1}} \leq \frac{1 - x}{\varphi^2(x) + n^{-1}}. \tag{2.36}$$

Further $(1 - (t + u_1 + \cdots + u_{2r})) \leq 1 - u$ and $1 - u \leq (\varphi^2(u)/\varphi^2(x))(1 - x)$ leads to the inequality

$$\varphi^2(x)\big(1 - (t + u_1 + \cdots + u_{2r})\big) \leq \varphi^2(u)(1 - x). \tag{2.37}$$

As $(t + u_1 + \cdots + u_{2r})(x - u) \geq x - u$ we get

$$(t + u_1 + \cdots + u_{2r}) - (t + u_1 + \cdots + u_{2r})x - u + ux$$
$$\leq (t + u_1 + \cdots + u_{2r}) - (t + u_1 + \cdots + u_{2r})u - x + ux,$$

and therefore

$$\frac{t + u_1 + \cdots + u_{2r} - u}{1 - u} \leq \frac{t + u_1 + \cdots + u_{2r} - x}{1 - x}. \tag{2.38}$$

Combining (2.38) and (2.36), we have

$$\begin{aligned}(t + u_1 + \cdots + u_{2r} - u)^{2r+m-1} &\leq \left(\frac{t + u_1 + \cdots + u_{2r} - x}{1 - x}(1 - u)\right)^{2r+m-1}\\ &\leq (t + u_1 + \cdots + u_{2r} - x)^{2r+m-1}\\ &\quad\times \left(\frac{\varphi^2(u) + n^{-1}}{\varphi^2(x) + n^{-1}}\right)^{m/2} \left(\frac{1 - u}{1 - x}\right)^{2r+(m/2)-1},\end{aligned}$$

and finally for $2r + (m/2) \geq 1$, i.e. $m \in \mathbb{N}$, and (2.37) one gets

$$\begin{aligned}&\varphi^{2r}(x)(t + u_1 + \cdots + u_{2r} - u)^{2r+m-1}\\ &\quad\leq \frac{(t + u_1 + \cdots + u_{2r} - x)^{2r+m-1}}{(\varphi^2(x) + n^{-1})^{m/2}} \varphi^{2r}(u)(\varphi^2(u) + n^{-1})^{m/2}\\ &\quad\quad\times \frac{(1 - x)^r}{(1 - (t + u_1 + \cdots + u_{2r}))^r}.\end{aligned} \tag{2.39}$$

So we have in both cases similar estimates. Using the Hardy–Littlewood majorant

$$G(\eta; x) := \sup_{t \neq x} \left| \frac{1}{x - t} \int_t^x |\eta(u)|\, du \right| \tag{2.40}$$

with

$$\eta(u) := \varphi^{2r}(u)(\varphi^2(u) + n^{-1})^{m/2} |g^{(2r+m)}(u)|,$$

we have for $p > 1$ by combining (2.35), (2.39), (2.40), applying the

Cauchy–Schwarz inequality together with (2.9) and Corollary 2.10, we obtain in Case 1

$$\begin{aligned}
&\left\| \varphi^{2r} K_n^{(2r)}(R_{2r+m} g) \right\|_p^{E_n} \\
&\quad \le C \left\{ \int_{E_n} \left| (\varphi^2(x) + n^{-1})^{-m/2} G(\eta; x) K_n^{(2r)} \right. \right. \\
&\qquad\qquad \left. \left. \times \left((x-t)^{2r+m}; x \right)_{u=x} \right|^p dx \right\}^{1/p} \\
&\quad \le C n^{-m/2} \|G\|_p \le C n^{-m/2} \left\| \varphi^{2r} (\varphi^2 + n^{-1})^{m/2} g^{(2r+m)} \right\|_p,
\end{aligned}$$

and in Case 2

$$\begin{aligned}
&\left\| \varphi^{2r} K_n^{(2r)}(R_{2r+m} g) \right\|_p^{E_n} \\
&\quad \le C \left\{ \int_{E_n} |G(\eta; x)|^p \left| \frac{(1-x)^r}{(\varphi^2(x) + n^{-1})^{m/2}} \left(K_{n,2r}\left((t-x)^{2m}; x \right) \right)^{1/2} \right. \right. \\
&\qquad\qquad \left. \left. \times \left(K_{n,2r}\left((1-t)^{-2r}; x \right) \right)^{1/2} \right|^p dx \right\}^{1/p} \\
&\quad \le C n^{-m/2} \left\| \varphi^{2r} (\varphi^2 + n^{-1})^{m/2} g^{(2r+m)} \right\|_p,
\end{aligned}$$

which proves the lemma for $p > 1$.

For $p = 1$ one has to use some typical ideas which can be found in [3, Chap. 9, p. 145.]. By rather lengthy but straightforward calculations using extensively the symmetry properties in Lemma 2.7 we obtain the "key inequality"

$$\begin{aligned}
&\left\| \varphi^{2r} K_n^{(2r)}(R_{2(r+m)} g) \right\|_1^{E_n} \\
&\quad \le C \int_{E_n} \sum_{K=0}^{N} p_{N,K}(x) \frac{|(K/N) - x|^{2m-1} + N^{-2m+1}}{\varphi^{2m}(x)} \\
&\qquad \times \left| \int_x^{K^*/(N+1)} \left| \varphi^{2r+2m}(u) g^{(2(r+m))}(u) \right| du \right| dx
\end{aligned} \tag{2.41}$$

for $N, K \in \mathbb{N}$ and

$$\frac{K^*}{N+1} := \min\left(1, \frac{K+2r+1}{N+1}\right).$$

For details, see [11, Lemma 5.12]. ∎

3. Main Results

First we prove a global direct result on weighted simultaneous approximation:

Theorem 3.1. *Let* $f \in L_p[0,1]$ *with* $f^{(2r)} \in L_p[0,1]$, $1 \le p < \infty$, $\varphi(x) = \sqrt{x(1-x)}$. *Then*

$$\left\|\varphi^{2r}(K_n f - f)^{(2r)}\right\|_p \le C\left\{\omega_\varphi^2(f^{(2r)}; n^{-1/2})_{\varphi^{2r},p} + n^{-1}\|\varphi^{2r} f^{(2r)}\|_p\right\} \quad (3.1)$$

with a constant C *independent of* n.

Proof. Using the equivalence (1.4) with

$$\begin{aligned}\bar{K}_\varphi^2(f^{(2r)}; t^2)_{\varphi^{2r},p} &= \inf\left\{\left\|\varphi^{2r}(f-g)^{(2r)}\right\|_p + t^2\left\|\varphi^{2r+2}(g^{(2r)})^{(2)}\right\|_p + t^4\left\|\varphi^{2r}(g^{(2r)})^{(2)}\right\|_p;\right.\\ &\qquad\left.\varphi^{2r}g^{(2r)}, \varphi^{2r}g^{(2r+2)} \in L_p[0,1]\right\},\end{aligned}$$

it has to be shown that

$$\left\|\varphi^{2r}(K_n f - f)^{(2r)}\right\|_p \le C\left\{\bar{K}_\varphi^2(f^{(2r)}; n^{-1})_{\varphi^{2r},p} + n^{-1}\|\varphi^{2r} f^{(2r)}\|_p\right\}. \quad (3.2)$$

For every $g \in L_p[0,1]$ with $g^{(2r+2)} \in L_p[0,1]$ we obtain from (2.3) and (2.11)

$$\begin{aligned}\left\|\varphi^{2r}(K_n f - f)^{(2r)}\right\|_p &\le \left\|\varphi^{2r}[K_n(f-g)]^{(2r)}\right\|_p + \left\|\varphi^{2r}[K_{n,2r}g^{(2r)} - g^{(2r)}]\right\|_p\\ &\quad + \left\|\varphi^{2r}(f-g)^{(2r)}\right\|_p\\ &\le C\left\{\left\|\varphi^{2r}(f-g)^{(2r)}\right\|_p + \left\|\varphi^{2r}[K_{n,2r}g^{(2r)} - g^{(2r)}]\right\|_p\right\}. \quad (3.3)\end{aligned}$$

In the following we have to estimate the second term in (3.3) by

$$\|\varphi^{2r}(K_{n,2r}g^{(2r)} - g^{(2r)})\|_p \le C\{n^{-1}(\|\varphi^{2r}g^{(2r)}\|_p + \|\varphi^{2r+2}g^{(2r+2)}\|_p) + n^{-2}\|\varphi^{2r}g^{(2r+2)}\|_p\}. \quad (3.4)$$

Therefore we expand $g^{(2r)}$ by the Taylor formula

$$g^{(2r)}(t) = g^{(2r)}(x) + (t-x)g^{(2r+1)}(x) + R_2(g^{(2r)}; t; x)$$

with the integral remainder

$$R_2(g^{(2r)}; t; x) := \int_x^t (t-u)g^{(2r+2)}(u)\,du, \qquad x, t \in (0,1).$$

Renumbering (2.3) and Lemma 2.1, we obtain

$$\begin{aligned}
\varphi^{2r}(x)K_{n,2r}(g^{(2r)}; x) &= \varphi^{2r}(x)\{K_{n,2r}(\Omega_{0,x}; x)g^{(2r)}(x) + K_{n,2r}(\Omega_{1,x}; x)g^{(2r+1)}(x) \\
&\qquad + K_{n,2r}(R_2(g^{(2r)}; t; x); x)\} \\
&= \frac{n!}{(n-2r)!(n+1)^{2r}}\varphi^{2r}(x) \\
&\quad \times \left\{g^{(2r)}(x) + \frac{((2r+1)/2)(1-2x)}{n+1}g^{(2r+1)}(x)\right. \\
&\qquad \left. + K_{n,2r}(R_2(g^{(2r)}; t; x); x)\right\}.
\end{aligned}$$

If we take L_p-norms we have with $1 - (n!/(n-2r)!(n+1)^{2r}) \le C/n$ for some C

$$\|\varphi^{2r}(K_{n,2r}g^{(2r)} - g^{(2r)})\|_p \le C\{n^{-1}(\|\varphi^{2r}g^{(2r)}\|_p + \|\varphi^{2r+2}g^{(2r+2)}\|_p) + \|\varphi^{2r}K_{n,2r}(R_2 g^{(2r)})\|_p\}, \quad (3.5)$$

where we have used that

$$\|\varphi^{2r}g^{(2r+1)}\|_p \le C\{\|\varphi^{2r+2}g^{(2r+2)}\|_p + \|\varphi^{2r}g^{(2r)}\|_p\}, \quad (3.6)$$

(see [7, p. 239, $c = -1$] and [3, p. 169 (10.5.1)]).

Applying lemma 2.5 with $m = 2$, (3.3) and (3.5),

$$\begin{aligned}\left\|\varphi^{2r}(K_n f - f)^{(2r)}\right\|_p &\le C\Big\{\left\|\varphi^{2r}(f-g)^{(2r)}\right\|_p + n^{-1}\big(\|\varphi^{2r}g^{(2r)}\|_p + \|\varphi^{2r+2}g^{(2r+2)}\|_p\big) \\ &\qquad + n^{-1}\left\|\varphi^{2r}(\varphi^2 + n^{-1})g^{(2r+2)}\right\|_p\Big\} \\ &\le C\Big\{\left\|\varphi^{2r}(f-g)^{(2r)}\right\|_p + n^{-1}\|\varphi^{2r+2}g^{(2r+2)}\|_p + n^{-2}\|\varphi^{2r}g^{(2r+2)}\|_p \\ &\qquad + n^{-1}\|\varphi^{2r}g^{(2r)}\|_p\Big\}.\end{aligned}$$

In view of $\|\varphi^{2r}g^{(2r)}\|_p \le \|\varphi^{2r}(f-g)^{(2r)}\|_p + \|\varphi^{2r}f^{(2r)}\|_p$ and by taking the infimum over all g we derive

$$\left\|\varphi^{2r}(K_n f - f)^{(2r)}\right\|_p \le C\Big\{\bar{K}_\varphi^2(f^{(2r)}; n^{-1})_{\varphi^{2r},p} + n^{-1}\|\varphi^{2r}f^{(2r)}\|_p\Big\},$$

which proves the theorem. ∎

3.1. Modified Voronovskaja Theorem

The following crucial estimate enables us to prove a converse result, which includes the saturation class for the weighted simultaneous L_p-approximation by the method $(K_n)_{n\in\mathbb{N}}$. A first result in this direction goes back to Bernstein [2], who constructed with the help of Voronovskaja's theorem for the Bernstein operators B_n, $n \in \mathbb{N}$, a new sequence $(Q_n)_{n\in\mathbb{N}}$, with $Q_n(f; x) = B_n(f - (\varphi^2(x)/2n)f''; x)$.

It is known that we obtain an asymptotic acceleration of convergence by adding the supplementary term $(-(\varphi^2(x)/2n)f'')$.

This idea was applied by Ditzian and Zhou [4] for the classical Kantorovič operators and in a joined paper by Lupaş and the author (see [10, Thm. 3.4]) for the V_n-operators in the form $(V_n - (1/n)(D_T f)) - f$ with $D_T f := (1 - x^2)f'' - xf'$.

So now we prove for the special linear combination $K_n f - (1/2(n - 2r))\varphi^2(K_n' f)'$ the following result utilizing the third weighted modulus of smoothness.

Theorem 3.2. *Let $f \in L_p[0,1]$ with $f^{(2r)} \in L_p[0,1]$, $1 \le p < \infty$, $\varphi(x) = \sqrt{x(1-x)}$ and $r \in \mathbb{N}$. Then for n sufficiently large*

$$\left\|\frac{(n-2r)!(n+1)^{2r}}{n!}\varphi^{2r}\left((K_n f - f)^{(2r)} - \frac{1}{2(n-2r)}\left(\varphi^2 K_n' f\right)^{(2r+1)}\right)\right\|_p \le C\left\{\omega_\varphi^3(f^{(2r)}; n^{-1/2})_{\varphi^{2r},p} + n^{-1}\|\varphi^{2r}f^{(2r)}\|_p\right\}. \quad (3.7)$$

Proof. We expand the best approximating polynomial $g \in \Pi_{[\sqrt{n}]}$ to f by Taylor's formula

$$g(t) = \sum_{i=0}^{2r+3} g^{(i)}(u)\frac{(t-u)^i}{i!} + R_{2r+4}(g;t;u)$$

with the integral remainder

$$R_{2r+4}(g;t;u) = \frac{1}{(2r+3)!}\int_u^t (t-v)^{2r+3} g^{(2r+4)}(v)\,dv, \qquad u,t \in (0,1).$$

With Lemma 2.1 and

$$\begin{aligned}
&-\frac{1}{2(n+1)}\left(\varphi^2(x)g'(x)\right)^{(2r+1)} \\
&\quad = -\frac{1}{2(n+1)}\left(\varphi^2(x)g^{(2r+2)}(x) + (2r+1)(1-2x)g^{(2r+1)}(x)\right. \\
&\qquad \left. -2r(2r+1)g^{(2r)}(x)\right) \qquad (3.8)
\end{aligned}$$

we get

$$\begin{aligned}
&\frac{(n-2r)!(n+1)^{2r}}{n!}\left(K_n^{(2r)}(g;x) - g^{(2r)}(x)\right) \\
&\quad - \frac{1}{2(n+1)}\left(\varphi^2(x)g'(x)\right)^{(2r+1)} \\
&\quad = g^{(2r)}(x)\left\{1 - \frac{(n-2r)!(n+1)^{2r}}{n!} + \frac{r(2r+1)}{(n+1)}\right\} \\
&\qquad + g^{(2r+2)}(x)\left(\frac{((6r^2+7r+2)/12)) - (2r^2+3r+1)\varphi^2(x)}{(n+1)^2}\right) \\
&\qquad + \frac{(n-2r)!(n+1)^{2r}}{n!}\left(\frac{g^{(2r+3)}(x)}{(2r+3)!}K_n^{(2r)}\left((t-u)^{2r+3};x\right)_{u=x}\right. \\
&\qquad \left. + K_n^{(2r)}\left(R_{2r+4}(g;t;u);x\right)_{u=x}\right). \qquad (3.9)
\end{aligned}$$

Using for $r \in \mathbb{N}$ the asymptotic identity

$$\left(1 - \frac{(n-2r)!(n+1)^{2r}}{n!} + \frac{r(2r+1)}{(n+1)}\right) = \mathcal{O}(n^{-2}) \qquad (n \to \infty) \tag{3.10}$$

we have

$$\begin{aligned}
&\frac{(n-2r)!(n+1)^{2r}}{n!}\left(K_n^{(2r)}(g;x) - g^{(2r)}(x)\right) \\
&\quad - \frac{1}{2(n+1)}\left(\varphi^2(x)g'(x)\right)^{(2r+1)} \\
&\quad = g^{(2r)}(x)\mathcal{O}(n^{-2}) + g^{(2r+2)}(x) \\
&\quad \times \left(\frac{((6r^2+7r+2)/12) - (2r^2+3r+1)\varphi^2(x)}{(n+1)^2}\right) \\
&\quad + \frac{(n-2r)!(n+1)^{2r}}{n!}\left(\frac{g^{(2r+3)}(x)}{(2r+3)!}K_n^{(2r)}\left((t-u)^{2r+3};x\right)_{u=x}\right. \\
&\quad \left. + K_n^{(2r)}\left(R_{2r+4}(g;t;u);x\right)_{u=x}\right).
\end{aligned} \tag{3.11}$$

The last two terms on the right-hand side of (3.11) are estimated with Hölder's inequality, (2.5) and (2.9) for $m = 4$ by

$$\begin{aligned}
\left|K_n^{(2r)}\left((t-u)^{2r+3};x\right)\right)_{u=x}\right| &\le M\left|\left\{K_n^{(2r)}\left((t-u)^{2r+4};x\right)_{u=x}\right\}^{3/4}\right| \\
&\le C\frac{\varphi^3(x)}{n^{3/2}}.
\end{aligned} \tag{3.12}$$

Combining now (3.11), (3.12) and (2.18) for $m = 4$ we obtain

$$\begin{aligned}
&\left\|\frac{(n-2r)!(n+1)^{2r}}{n!}\varphi^{2r}\left\{K_n^{(2r)}g - g^{(2r)}\right\} - \frac{\varphi^{2r}}{2(n+1)}\left\{\varphi^2 g'\right\}^{(2r+1)}\right\|_p \\
&\quad \le C\left(n^{-2}\|\varphi^{2r}g^{(2r)}\|_p + n^{-2}\|\varphi^{2r}g^{(2r+2)}\|_p + n^{-2}\|\varphi^{2r+2}g^{(2r+2)}\|_p\right. \\
&\quad \left. + n^{-3/2}\|\varphi^{2r+3}g^{(2r+3)}\|_p + n^{-2}\|\varphi^{2r+4}g^{(2r+4)}\|_p\right).
\end{aligned} \tag{3.13}$$

The next step of our proof will be to prove the estimate

$$\left\| \frac{(n-2r)!(n+1)^{2r}}{n!} \left(\varphi^{2r}\{K_n^{(2r)}g - g^{(2r)}\} - \frac{\varphi^{2r}}{2(n-2r)}\{\varphi^2 K_n' g\}^{(2r+1)} \right) \right\|_p$$
$$\leq C\{\omega_\varphi^3(g^{(2r)}; n^{-1/2})_{\varphi^{2r},p} + n^{-1}\|\varphi^{2r}g^{(2r)}\|_p\}. \tag{3.14}$$

In order to do this we have to complete the missing parts by the following estimate

$$\left\| \frac{\varphi^{2r}}{2(n+1)}(\varphi^2 g')^{(2r+1)} - \frac{(n-2r)!(n+1)^{2r}}{n!} \frac{\varphi^{2r}}{2(n-2r)}\{\varphi^2 K_n' g\}^{(2r+1)} \right\|_p$$
$$\leq C\left[n^{-3/2}\|\varphi^{2r+3}g^{(2r+3)}\|_p + n^{-1}\|\varphi^{2r}g^{(2r)}\|_p + n^{-2}\|\varphi^{2r}g^{(2r+2)}\|_p\right]$$
$$=: C\mathcal{S}_n. \tag{3.15}$$

This is done by applying Leibniz's rule and verifying the following three estimates (3.16)–(3.18)

$$\left\| \frac{\varphi^{2r+2}}{2(n+1)} g^{(2r+2)} - \frac{(n-2r)!(n+1)^{2r}}{n!} \frac{\varphi^{r+2}}{2(n-2r)} K_n^{(2r+2)} g \right\|_p \leq C\mathcal{S}_n, \tag{3.16}$$

$$\left\| \frac{(1-2x)(2r+1)\varphi^{2r}}{2(n+1)} g^{(2r+1)} - \frac{(n-2r)!(n+1)^{2r}}{n!} \times \frac{(1-2x)(2r+1)\varphi^{2r}}{2(n-2r)} K_n^{(2r+1)} g \right\|_p \leq C\mathcal{S}_n, \tag{3.17}$$

and

$$\left\| \frac{(2r+1)r\varphi^{2r}}{n+1} g^{(2r)} - \frac{(n-2r)!(n+1)^{2r}}{n!} \frac{(2r+1)r\varphi^{2r}}{n-2r} K_n^{(2r)} g \right\|_p \leq C\mathcal{S}_n. \tag{3.18}$$

We expand again the best approximating polynomial g of degree $[\sqrt{n}]$ to f by Taylor's formula (in a different way than above)

$$g(t) = \sum_{i=0}^{2r+2} g^{(i)}(u) \frac{(t-u)^i}{i!} + R_{2r+3}(g; t; u)$$
$$=: P_{2r+2}(t) + R_{2r+3}(g; t; u) \tag{3.19}$$

with integral remainder

$$R_{2r+3}(g;t;u) = \frac{1}{(2r+2)!}\int_u^t (t-v)^{2r+2} g^{(2r+3)}(v)\,dv, \qquad u,t \in (0,1).$$

Using Lemma 2.1, the functions on the left-hand side of (3.16), (3.17), and (3.18), respectively are transformed into

$$\frac{r+1}{(n+1)^2}\varphi^{2r+2}(x)g^{(2r+2)}(x) - \frac{(n-2r)!(n+1)^{2r}}{n!}\frac{\varphi^{2r+2}(x)}{2(n-2r)}K_n^{(2r+2)}(R_{2r+3}(g;t;u);x)_{u=x}, \tag{3.20}$$

$$\left(\frac{2\varphi^{2r+2}(x)(2r+1)(r+1)}{(n+1)^2} - \frac{\varphi^{2r}(x)(2r+1)(r+1)}{2(n+1)^2}\right)g^{(2r+2)}(x) - \frac{(n-2r)!(n+1)^{2r}}{n!}\frac{\varphi^{2r}(x)(1-2x)(2r+1)}{2(n-2r)}K_n^{(2r+1)} \times (R_{2r+3}(g;t;u);x)_{u=x} \tag{3.21}$$

and

$$(2r+1)r\varphi^{2r}(x)\left(-\frac{(2r+1)}{(n+1)(n-2r)}g^{(2r)}(x) - \frac{(2r+1)(1-2x)}{2(n+1)(n-2r)}g^{(2r+1)}(x) - \frac{((3r+2)(2r+1)/6) - (4r^2+6r-n+1)\varphi^2(x)}{2(n-2r)(n+1)^2}g^{(2r+2)}(x) - \frac{(n-2r)!(n+1)^{2r}}{n!(n-2r)}K_n^{(2r)}(R_{2r+3}(g;t;u);x)_{u=x}\right). \tag{3.22}$$

Now the terms containing the integral remainder in those expressions can be estimated in the following way:

$$\left|\frac{(n-2r)!(n+1)^{2r}}{2(n-2r)n!}\varphi^{2r+2}(x)K_n^{(2r+2)}(R_{2r+3}(g;t;u);x)_{u=x}\right|$$
$$\leq Cn^{-1}\left|\varphi^{2r+2}(x)K_n^{(2r+2)}(R_{2r+3}(g;t;u);x)_{u=x}\right|, \quad (3.23)$$

$$\left|\frac{(n-2r)!(n+1)^{2r}(2r+1)(1-2x)}{2(n-2r)n!}\varphi^{2r}(x)K_n^{(2r+1)}\right.$$
$$\left.\times(R_{2r+3}(g;t;u);x)_{u=x}\right|$$
$$\leq Cn^{-1}\left|(1-2x)\varphi^{2r}(x)K_n^{(2r+1)}(R_{2r+3}(g;t;u);x)_{u=x}\right|, \quad (3.24)$$

$$\left|\frac{(n-2r)!(n+1)^{2r}(2r+1)r}{(n-2r)n!}\varphi^{2r}(x)K_n^{(2r)}(R_{2r+3}(g;t;u);x)_{u=x}\right|$$
$$\leq Cn^{-1}\left|\varphi^{2r}(x)K_n^{(2r)}(R_{2r+3}(g;t;u);x)_{u=x}\right|, \quad (3.25)$$

respectively.

With (2.11) and P_{2r+2} from (3.19) for g we derive

$$\|\varphi^{2r}K_n^{(2r)}R_{2r+3}g\|_p \leq \|\varphi^{2r}K_n^{(2r)}g\|_p + \|\varphi^{2r}K_n^{(2r)}P_{2r+2}\|_p$$
$$\leq C\left\{\|\varphi^{2r}g^{(2r)}\|_p + \|\varphi^{2r}K_n^{(2r)}P_{2r+2}\|_p\right\}.$$

Applying Lemma 2.1 once more, we get

$$\varphi^{2r}(x)K_n^{(2r)}(P_{2r+2};x)_{u=x}$$
$$= \varphi^{2r}(x)\sum_{i=0}^{2r+2}\frac{g^{(i)}(x)}{i!}K_n^{(2r)}((t-u)^i;x)_{u=x}$$
$$= \varphi^{2r}(x)\frac{n!}{(n-2r)!(n+1)^{2r}}$$
$$\times\left(g^{(2r)}(x) + g^{(2r+1)}(x)\frac{(2r+1)(1-2x)}{2(n+1)}\right.$$
$$\left.+g^{(2r+2)}(x)\frac{((3r+2)(2r+1)/6)-(4r^2+6r-n+1)\varphi^2(x)}{2(n+1)^2}\right)$$
$$\leq C\{\varphi^{2r}(x)g^{(2r)}(x) + n^{-1}\varphi^{2r}(x)g^{(2r+1)}(x)$$
$$+n^{-2}\varphi^{2r}(x)g^{(2r+2)}(x) + n^{-1}\varphi^{2r+2}(x)g^{(2r+2)}(x)\}$$

and with

$$\|\varphi^{2r}K_n^{(2r)}P_{2r+2}\|_p \le C\{\|\varphi^{2r}g^{(2r)}\|_p + n^{-1}\|\varphi^{2r}g^{(2r+1)}\|_p + n^{-2}\|\varphi^{2r}g^{(2r+2)}\|_p + n^{-1}\|\varphi^{2r+2}g^{(2r+2)}\|_p\}$$

we have in (3.25) the upper bound

$$n^{-1}\|\varphi^{2r}K_n^{(2r)}R_{2r+3}g\|_p \le C\{n^{-1}\|\varphi^{2r}g^{(2r)}\|_p + n^{-2}(\|\varphi^{2r}g^{(2r+1)}\|_p + \|\varphi^{2r+2}g^{(2r+2)}\|_p) + n^{-3}\|\varphi^{2r}g^{(2r+2)}\|_p\}.$$

It is sufficient now to obtain an upper bound for (3.23). Since on account of (3.6) the upper bound for (3.24) is a combination of (3.23) and (3.25).

We have with (2.19) for $m = 1$ and $p = 1$

$$\|\varphi^{2r+2}K_n^{(2r+2)}R_{2r+3}g\|_1 \le Cn^{-1/2}\|\varphi^{2r+3}g^{(2r+3)}\|_1$$

and with $(\varphi^2(x) + n^{-1})^{1/2} \le \varphi(x) + n^{-1/2}$ for $p > 1$

$$\|\varphi^{2r+2}K_n^{(2r+2)}R_{2r+3}g\|_p \le Cn^{-1/2}\|\varphi^{2r+2}(\varphi^2 + n^{-1})^{1/2}g^{(2r+3)}\|_p \le C\{n^{-1/2}\|\varphi^{2r+3}g^{(2r+3)}\|_p + n^{-1}\|\varphi^{2r+2}g^{(2r+3)}\|_p\}.$$

So we get for all $p \ge 1$

$$n^{-1}\|\varphi^{2r+2}K_n^{(2r+2)}R_{2r+3}g\|_p \le C\{n^{-3/2}\|\varphi^{2r+3}g^{(2r+3)}\|_p + n^{-2}\|\varphi^{2r+2}g^{(2r+3)}\|_p\},$$

which proves the inequality (3.15).

Finally, with (3.13) and (3.15) there is

$$\left\|\frac{(n-2r)!(n+1)^{2r}}{n!}\left\{\varphi^{2r}[K_n^{(2r)}g - g^{(2r)}] - \frac{\varphi^{2r}}{2(n-2r)}[\varphi^2 K_n' g]^{(2r+1)}\right\}\right\|_p \le C\{n^{-1}\|\varphi^{2r}g^{(2r)}\|_p + n^{-3/2}\|\varphi^{2r+3}g^{(2r+3)}\|_p + n^{-2}(\|\varphi^{2r}g^{(2r)}\|_p + \|\varphi^{2r}g^{(2r+2)}\|_p + \|\varphi^{2r+2}g^{(2r+2)}\|_p + \|\varphi^{2r+4}g^{(2r+4)}\|_p)\}. \quad (3.26)$$

The following steps give estimates for the terms of the right-hand side of (3.26) involving the third order weighted modulus of smoothness $\omega_\varphi^3(f^{(2r)}; n^{-1/2})_{\varphi^{2r},p}$.

First of all we get

$$n^{-3/2}\|\varphi^{2r+3}g^{(2r+3)}\|_p \le n^{-3/2}\|\varphi^{2r+3}g^{(2r+3)}\|_p + \left\|\varphi^{2r}\left(f^{(2r)} - g^{(2r)}\right)\right\|_p$$
$$\le CK_\varphi^3\left(f^{(2r)}; \left(n^{-1/2}\right)^3\right)_{\varphi^{2r},p}$$
$$\le C\omega_\varphi^3\left(f^{(2r)}; n^{-1/2}\right)_{\varphi^{2r},p},$$

and with [3, p. 59]

$$n^{-2}\|\varphi^{2r+4}g^{(2r+4)}\|_p \le n^{-2}\|\varphi^{2r+4}g^{(2r+4)}\|_p + \left\|\varphi^{2r}\left(f^{(2r)} - g^{(2r)}\right)\right\|_p$$
$$\le CK_\varphi^4\left(f^{(2r)}; \left(n^{-1/2}\right)^4\right)_{\varphi^{2r},p}$$
$$\le C\omega_\varphi^4\left(f^{(2r)}; n^{-1/2}\right)_{\varphi^{2r},p} \le C\omega_\varphi^3\left(f^{(2r)}; n^{-1/2}\right)_{\varphi^{2r},p}.$$

Next with (3.6)

$$n^{-2}\|\varphi^{2r}g^{(2r+2)}\|_p$$
$$\le Cn^{-2}\left\{\|\varphi^{2r+2}g^{(2r+3)}\|_p + \|\varphi^{2r}g^{(2r+1)}\|_p\right\}$$
$$\le Cn^{-2}\left\{\|\varphi^{2r+2}g^{(2r+3)}\|_p + \|\varphi^{2r+2}g^{(2r+2)}\|_p + \|\varphi^{2r}g^{(2r)}\|_p\right\},$$

and together with $\varphi(x) \sim n^{-1/2}$ and two inequalities of the weighted approximation by polynomials (see [3, p. 91 and p. 108]) we have

$$n^{-2}\|\varphi^{2r+2}g^{(2r+3)}\|_p = n^{-3/2}\|n^{-1/2}\varphi^{2r+2}g^{(2r+3)}\|_p$$
$$\le Cn^{-3/2}\|n^{-1/2}\varphi^{2r+2}g^{(2r+3)}\|_p^{[1/n,1-(1/n)]}$$
$$\le Cn^{-3/2}\|\varphi^{2r+3}g^{(2r+3)}\|_p^{[1/n,1-(1/n)]}$$
$$\le Cn^{-3/2}\|\varphi^{2r+3}g^{(2r+3)}\|_p$$

and

$$n^{-2}\|\varphi^{2r+2}g^{(2r+2)}\|_p \le Cn^{-1}\|\varphi^{2r}g^{(2r)}\|_p.$$

Finally we get with this consideration for (3.26) that

$$\left\|\frac{(n-2r)!(n+1)^{2r}}{n!}\left\{\varphi^{2r}\left[K_n^{(2r)}g - g^{(2r)}\right] - \frac{\varphi^{2r}}{2(n-2r)}\left[\varphi^2 K_n' g\right]^{(2r+1)}\right\}\right\|_p$$
$$\le C\omega_\varphi^3\left(f^{(2r)}; n^{-1/2}\right)_{\varphi^{2r},p} \cdot + n^{-1}\|\varphi^{2r}g^{(2r)}\|_p, \tag{3.27}$$

which is already the desired result for the polynomial of best approximation to f. Now we make use of the fact that $f = (f-g) + g =: F + g$

with $\|\varphi^{2r}(f-g)^{(2r)}\|_p \le C\omega_\varphi^3(f^{(2r)}; n^{-1/2})_{\varphi^{2r},p}$ [3, p. 55]. On the one hand, we obtain with (2.11)

$$\left\| \frac{(n-2r)!(n+1)^{2r}}{n!} \varphi^{2r} K_n^{(2r)} F \right\|_p \le C\|\varphi^{2r} F^{(2r)}\|_p,$$

and, on the other hand, we still have to show that

$$\left\| \frac{(n-2r)!(n+1)^{2r}}{n!} \frac{\varphi^{2r}}{2(n-2r)} \left(\varphi^2 K_n' F\right)^{(2r+1)} \right\|_p$$
$$\le C\|\varphi^{2r} F^{(2r)}\|_p = C\left\|\varphi^{2r}(f-g)^{(2r)}\right\|_p$$

which means in detail that we have to verify the estimate

$$\left\| \frac{(n-2r)!(n+1)^{2r}}{n!2(n-2r)} \left(\varphi^{2r+2} K_n^{(2r+2)} F + (2r+1)(1-2x)\varphi^{2r} K_n^{(2r+1)} F \right.\right.$$
$$\left.\left. -2r(2r+1)\varphi^{2r} K_n^{(2r)} F\right) \right\|_p \le C\|\varphi^{2r} F^{(2r)}\|_p.$$

This is done by using ((2.13), $s = 1$), which gives

$$\left\| \frac{(n-2r)!(n+1)^{2r}}{n!} \frac{\varphi^{2r+2}}{2(n-2r)} K_n^{(2r+2)} F \right\|_p$$
$$\le C \frac{(n-2r)!(n+1)^{2r}}{n!} \frac{n}{2(n-2r)} \|\varphi^{2r} F^{(2r)}\|_p$$
$$\le C\|\varphi^{2r} F^{(2r)}\|_p,$$

and (2.11); we obtain

$$\left\| \frac{(n-2r)!(n+1)^{2r}}{n!} \frac{\varphi^{2r}}{2(n-2r)} K_n^{(2r)} F \right\|_p$$
$$\le C \frac{(n-2r)!(n+1)^{2r}}{n!2(n-2r)} \|\varphi^{2r} F^{(2r)}\|_p$$
$$\le \frac{C}{n-2r} \|\varphi^{2r} F^{(2r)}\|_p \le C\|\varphi^{2r} F^{(2r)}\|_p.$$

Also for the last term we estimate

$$\begin{aligned}
&\left\| \frac{(n-2r)!(n+1)^{2r}}{n!} \frac{\varphi^{2r}}{2(n-2r)} K_n^{(2r+1)} F \right\|_p \\
&\le C \frac{(n-2r)!(n+1)^{2r}}{n!} \frac{1}{2(n-2r)} \\
&\quad \times \left\{ \|\varphi^{2r+2} K_n^{(2r+2)} F\|_p + \|\varphi^{2r} K_n^{(2r)} F\|_p \right\} \\
&\le C \|\varphi^{2r} F^{(2r)}\|_p
\end{aligned}$$

and finally we have with (3.27) for $p \ge 1$

$$\begin{aligned}
&\left\| \frac{(n-2r)!(n+1)^{2r}}{n!} \left(\varphi^{2r}(K_n f - f)^{(2r)} - \frac{\varphi^{2r}}{2(n-2r)} (\varphi^2 K_n' f)^{(2r+1)} \right) \right\|_p \\
&\le \left\| \frac{(n-2r)!(n+1)^{2r}}{n!} \right. \\
&\qquad \left. \times \left(\varphi^{2r}(K_n F - F)^{(2r)} - \frac{\varphi^{2r}}{2(n-2r)} (\varphi^2 K_n' F)^{(2r+1)} \right) \right\|_p \\
&\quad + \left\| \frac{(n-2r)!(n+1)^{2r}}{n!} \right. \\
&\qquad \left. \times \left(\varphi^{2r}(K_n g - g)^{(2r)} - \frac{\varphi^{2r}}{2(n-2r)} (\varphi^2 K_n' g)^{(2r+1)} \right) \right\|_p \\
&\le C \left\{ \omega_\varphi^3(f^{(2r)}; n^{-1/2})_{\varphi^{2r}, p} + n^{-1} \|\varphi^{2r} f^{(2r)}\|_p \right\},
\end{aligned}$$

which proves the theorem. ∎

3.2. The Equivalence Result

Now it is the aim of this subsection to show a complete characterization result, which includes an extensive class of functions.

Up to this point we can show with Theorem 3.1 and the Berens–Lorentz lemma [1, p. 694] for $f \in L_p[0,1]$ with $f^{(2r)} \in L_p[0,1]$, $1 \le p \le \infty$ the equivalence between the following two statements

$$\left\|\varphi^{2r}(K_n f - f)^{(2r)}\right\|_p = \mathscr{O}(n^{r-\alpha}), \qquad (n \to \infty), \tag{3.28}$$

$$\omega_\varphi^2(f^{(2r)}, t)_{\varphi^{2r}, p} = \mathscr{O}(t^{2(\alpha - r)}), \qquad (t \to 0), \tag{3.29}$$

for $r \in \mathbb{N}$ and $r < \alpha < r + 1$.

This subsection will extend the above result and close the gap for the case $\alpha = r + 1$. The key idea for this extension is to replace in all estimates the weighted modulus of smoothness by a so-called order function ψ which is a nonnegative function defined on $[0, 1]$ with the property

$$\psi(kt) \le Ck^2\psi(t), \tag{3.30}$$

for $t > 0$, $k \in \mathbb{N}$ (C a suitable constant independent of t).

Examples are the functions $\psi(t) := t^\alpha$, $\alpha \in (0, 2]$ and $\psi(t) := t^2|\ln t|^\beta$, $\beta \in \mathbb{R}$.

The following lemma [20, p. 259] shows that an order function ψ has similar properties as the weighted modulus of smoothness.

LEMMA 3.3. *Let ψ_1, ψ_2 be monotone increasing and nonnegative functions defined on $[0, 1]$. If*

$$\psi_1(t) \le C\left\{\psi_2(h) + \frac{t^r}{h^r}\psi_1(h)\right\}, \qquad \text{for all } 0 < t, h \le 1, r > 0, \tag{3.31}$$

then

$$\psi_1(h) \le A\left\{h^{r-(1/2)}\int_h^1 \frac{\psi_2(t)}{t^{r+(1/2)}}\, dt + h^{r-(1/2)}\right\}, \tag{3.32}$$

where A depends on $C > 1$ and $\psi_1(1), \psi_2(1)$.

THEOREM 3.4 (Equivalence Result). *Let $f \in L_p[0, 1]$ with $f^{(2r)} \in L_p[0, 1]$, $1 \le p < \infty$, $r \in \mathbb{N}$, $\varphi(x) = \sqrt{x(1 - x)}$ and ψ be an order function satisfying (3.30). Then the following statements are equivalent:*

(i) $\left\|\varphi^{2r}(K_n f - f)^{(2r)}\right\|_p = \mathscr{O}\left(\psi(n^{-1/2}) + \frac{1}{n}\right) \qquad (n \to \infty);$

(ii) $\omega_\varphi^2(f^{(2r)}; t)_{\varphi^{2r}, p} = \mathscr{O}(\psi(t) + t^2) \qquad (t \to 0).$

We have for the functions $\psi(t) = t^2$ and $\psi(t) = t^2|\ln t|$ the following corollary which closes the gap in the first equivalence result (3.28) and (3.29) and further it gives us a complete characterization form with the "intermediate regulation" functions for the rate of weighted simultaneous L_p-approximation.

COROLLARY 3.5. *Let* $f \in L_p[0,1]$ *with* $f^{(2r)} \in L_p[0,1]$, $1 \le p < \infty$, $r \in \mathbb{N}$, *and* $\varphi(x) = \sqrt{x(1-x)}$.

1. *Especially for* $\psi(t) = t^2$ *we obtain the equivalence*

$$\begin{aligned} \left\|\varphi^{2r}(K_n f - f)^{(2r)}\right\|_p &= \mathcal{O}(n^{-1}) \qquad (n \to \infty), \\ \Leftrightarrow \omega_\varphi^2(f^{(2r)};t)_{\varphi^{2r},p} &= \mathcal{O}(t^2) \qquad (t \to 0), \\ \Leftrightarrow \omega_\varphi^{2r+2}(f;t)_p &= \mathcal{O}(t^{2r+2}) \qquad (t \to 0), \end{aligned}$$

closing the above mentioned gap.

2. *For* $\psi(t) = t^2|\ln t|$ *we obtain the new equivalence*

$$\begin{aligned} \left\|\varphi^{2r}(K_n f - f)^{(2r)}\right\|_p &= \mathcal{O}(n^{-1}|\ln n|) \qquad (n \to \infty), \\ \Leftrightarrow \omega_\varphi^2(f^{(2r)};t)_{\varphi^{2r},p} &= \mathcal{O}(t^2|\ln t|) \qquad (t \to 0), \\ \Leftrightarrow \omega_\varphi^{2r+2}(f;t)_p &= \mathcal{O}(t^{2(r+1)}|\ln t|) \qquad (t \to 0). \end{aligned}$$

Proof of theorem 3.4. For the direct result (ii) $\Rightarrow$ (i) we see that with Theorem 3.1

$$\begin{aligned} \left\|\varphi^{2r}(K_n f - f)^{(2r)}\right\|_p &\le C\left\{\omega_\varphi^2(f^{(2r)};n^{-1/2})_{\varphi^{2r},p} + n^{-1}\|\varphi^{2r}f^{(2r)}\|_p\right\} \\ &= \mathcal{O}\left(\psi(n^{-1/2}) + \frac{1}{n}\right). \end{aligned}$$

On the other hand, for (i) $\Rightarrow$ (ii) we have to prove only the important case for $\psi(t) \ge Ct^2$ that means that we have the assumption

$$\left\|\varphi^{2r}(K_n f - f)^{(2r)}\right\|_p = \mathcal{O}(\psi(n^{-1/2})). \tag{3.33}$$

In a first consideration we show that $\omega_\varphi^4(f^{(2r)};t)_{\varphi^{2r},p}$ satisfies the assumption of Lemma 3.3.

Lemma 2.2 (for $s = 2$) and the monotonicity of the modulus of smoothness gives for g with $\varphi^{2r}g^{(2r)}, \varphi^{2r+4}g^{(2r+4)} \in L_p[0,1]$ and $t \sim n^{-1/2}$, $t > 0$

$$\begin{aligned} &\omega_\varphi^4(f^{(2r)};t)_{\varphi^{2r},p} \\ &\quad \le C\Big\{\left\|\varphi^{2r}(f - K_n f)^{(2r)}\right\|_p + t^4\Big(\left\|\varphi^{2r+4}K_n^{(2r+4)}(f-g)\right\|_p \\ &\qquad\qquad + \|\varphi^{2r+4}K_n^{(2r+4)}g\|_p\Big)\Big\} \\ &\quad \le C\Big\{\psi(n^{-1/2}) + t^4 n^2\Big(\left\|\varphi^{2r}(f-g)^{(2r)}\right\|_p + n^{-2}\|\varphi^{2r+4}g^{(2r+4)}\|_p\Big)\Big\}. \end{aligned}$$

Taking the infimum over those g and making use of the equivalence between weighted K-functionals and corresponding weighted moduli of smoothness, we obtain

$$\omega_\varphi^4(f^{(2r)};t)_{\varphi^{2r},p} \leq C\big(\psi(n^{-1/2}) + t^4 n^2 \omega_\varphi^4(f^{(2r)};n^{-1/2})_{\varphi^{2r},p}\big). \quad (3.34)$$

This upper bound of $\omega_\varphi^4(f^{(2r)};t)_{\varphi^{2r},p}$ corresponds to the inequality (3.31) in Lemma 3.3 with $r = 4$ and $h = n^{-1/2}$.

Combining this and property (3.30) for ψ we see that $\omega_\varphi^4(f^{(2r)};t)_{\varphi^{2r},p}$ behaves similar as the function ψ_1 in Lemma 3.3. So we get with (3.34)

$$\begin{aligned}\omega_\varphi^4(f^{(2r)};h)_{\varphi^{2r},p} &\leq Ah^{4-(1/2)}\int_h^1 \frac{\psi(t)}{t^{4+(1/2)}}\,dt \\ &\leq \frac{2}{3}CA(1-h^{3/2})\psi(h) \leq C\psi(h). \quad (3.35)\end{aligned}$$

Using Marchaud's inequality [3, Chap. 6] there is

$$\omega_\varphi^3(f^{(2r)};h)_{\varphi^{2r},p} \leq C\psi(h).$$

With the modified Voronovskaja Theorem 3.2 we know that for $f \in L_p[0,1]$ with $f^{(2r)} \in L_p[0,1]$ the estimate

$$\begin{aligned}&\left\| \frac{(n-2r)!(n+1)^{2r}}{n!}\varphi^{2r}\left((K_n f - f)^{(2r)} - \frac{1}{2(n-2r)}(\varphi^2 K_n' f)^{(2r+1)}\right)\right\|_p \\ &\quad\leq C\left\{\omega_\varphi^3(f^{(2r)};n^{-1/2})_{\varphi^{2r},p} + n^{-1}\|\varphi^{2r}f^{(2r)}\|_p\right\} \leq C\psi(n^{-1/2})\end{aligned}$$

holds true.

From this and the assumption (3.33) we get

$$\begin{aligned}&\left\|\frac{\varphi^{2r}}{2(n-2r)}(\varphi^2 K_n' f)^{(2r+1)}\right\|_p \\ &\quad= \left\|\frac{\varphi^{2r+2}}{2(n-2r)}K_n^{(2r+2)}f + \frac{(2r+1)(1-2x)}{2(n-2r)}\varphi^{2r}K_n^{(2r+1)}f \right.\\ &\qquad\qquad \left. - \frac{2r+1}{n-2r}\varphi^{2r}K_n^{(2r)}f\right\|_p \\ &\quad\leq C\psi(n^{-1/2}). \quad (3.36)\end{aligned}$$

To complete the proof we still have to verify the estimate

$$\left\| \frac{1}{2(n-2r)} \varphi^{2r+2} K_n^{(2r+2)} f \right\|_p \leq C\psi(n^{-1/2}), \tag{3.37}$$

since then

$$\begin{aligned} K_\varphi^2(f^{(2r)}; t^2)_{\varphi^{2r},p} &\leq C\left\{ \left\| \varphi^{2r}(f - K_n f)^{(2r)} \right\|_p + t^2 \|\varphi^{2r+2} K_n^{(2r+2)} f\|_p \right\} \\ &\leq C\psi(t) \leq C(\psi(t) + t^2) \end{aligned}$$

and the final conclusion (i) $\Rightarrow$ (ii) of this theorem is proved.

First of all we obtain that (3.36) and the Bernstein inequality (2.11) imply

$$\left\| \frac{\varphi^{2r+2}}{2(n-2r)} K_n^{(2r+2)} f + \frac{(2r+1)(1-2x)}{2(n-2r)} \varphi^{2r} K_n^{(2r+1)} f \right\|_p \leq C\psi(n^{-1/2}). \tag{3.38}$$

If

$$\frac{1}{2(n-2r)} \|\varphi^{2r} K_n^{(2r+1)} f\|_p \leq C\psi(n^{-1/2})$$

then we get with (3.38) the estimate (3.37).

This is done for $p > 1$ in a routine way utilizing the Hardy–Littlewood majorant of the denominator on the left-hand side of (3.38) multiplied with φ^{2r}.

In the missing case $p = 1$ we have for $0 \leq x \leq \frac{1}{2}$:

$$\begin{aligned} \|\varphi^{2r} K_n^{(2r+1)} f\|_1^{[0,1/2]} &= \int_0^{1/2} \frac{1}{\varphi^{2r+2}(x)} \left| \int_0^x H(t) \varphi^{2r}(t)\, dt \right| dx \\ &\leq \int_0^{1/2} |H(t)| \varphi^{2r}(t) \int_t^{1/2} \varphi^{-2r-2}(x)\, dx\, dt \\ &\leq 2^{r+1} \int_0^{1/2} |H(t)| \varphi^{2r}(t) \int_t^{1/2} x^{-r-1}\, dx\, dt \\ &\leq C\|H\|_1. \end{aligned}$$

With similar considerations we have for $x \in [\frac{1}{2}, 1]$ and $\frac{1}{2}(1-x) \leq \varphi^2(x) \leq (1-x)$ that $\|\varphi^{2r} K_n^{(2r+1)} f\|_1^{[1/2,1]} \leq C\|H\|_1$.

Finally we get for $1 \leq p < \infty$

$$\|\varphi^{2r} K_n^{(2r+1)} f\|_p \leq C\|H\|_p,$$

together with (3.38)

$$\frac{1}{2(n-2r)}\|\varphi^{2r} K_n^{(2r+1)} f\|_p \leq C\left\|\frac{1}{2(n-2r)} H\right\|_p \leq C\psi(n^{-1/2}),$$

which proves the theorem. ▮

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